MORE PATHOLOGIES OF THE VOLUME FUNCTION

ABSTRACT. Suppose that D is a pseudoeffective $\mathbb{R}$-divisor on a smooth projective variety X, and fix an ample $\mathbb{R}$-divisor A. It has often proved useful to higher-dimensional geometry to study the growth of $h^0(X, \lfloor mD \rfloor + A)$ as m increases; these provide a numerically-invariant analog of the Iitaka dimension. Previous work of the second author showed that this growth, somewhat unexpectedly, might not be even approximately polynomial in m. In this note, we show that even worse behavior is possible: we describe examples in which this oscillates between two different powers in m, and in which the exponent is an irrational number. The results are based on an analysis of the volume function on a certain Calabi–Yau threefold with a large group of pseudoautomorphisms. The analysis is based on the study nearly rational geodesics on a hyperbolic three-manifold.

1. INTRODUCTION

Suppose that X is a projective variety over an algebraically closed field, and that D is a Cartier divisor on X. It is a fundamental result that if one considers the spaces $H^0(X, mD)$, the dimension grows more or less polynomially in m:

Theorem 1. *Suppose that X is a smooth projective variety over K and that D is a line bundle on X. There exist constants $C_1, C_2 > 0$ and an integer κ so that for all sufficiently large and divisible m,*

$$C_1 m^{\kappa} < h^0(X, mD) < C_2 m^{\kappa}.$$

It is sometimes convenient to consider a version of this with additional twisting by an ample divisor: if D is a pseudoeffective $\mathbb{R}$-divisor, one can fix an ample A and examine the growth of $h^0(mD + A)$ as m increases. This variant plays a crucial role in [BCHM], for example.

It was initially hoped that this growth might also be polynomial in m [**?**, **?**, **?**]. However, a recent example shows that this is not the case in general, at least if one allows the definition to be extended to the set of $\mathbb{R}$-divisors.

Theorem 2. *Let X be a complete intersection of type $(1, 1)$, $(1, 1)$, $(2, 2)$ in $\mathbb{P}^3 \times \mathbb{P}^3$. There exists a pseudoeffective $\mathbb{R}$-divisor D on X such that for any sufficiently ample A, there are constants $C_1, C_2 > 0$ so that*

$$C_1 m^{3/2} < h^0(\lfloor mD \rfloor + A) < C_2 m^{3/2}$$

Although this growth rate is not polynomial, it is not too far removed, and it seems natural to ask what other kinds of growth are possible. Might the growth always resemble Cm^q for a rational number q, or perhaps a real number q?

In this note we show that even this is too much to hope for. Through an analysis of the volume function on a certainly Calabi–Yau threefold, we show that the volume function can exhibit a sort of oscillatory behavior.

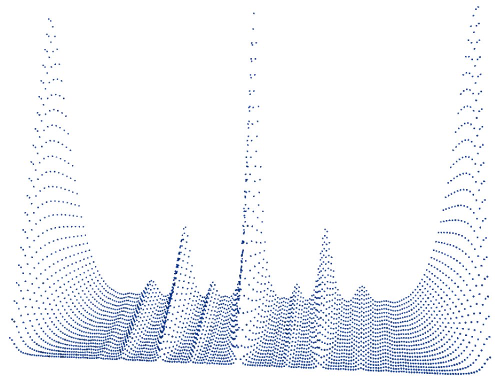

FIGURE 1. Volume on the Wehler 3-fold

Theorem 3. *There exists a smooth threefold X with an $\mathbb{R}$-divisor D so that*

$$\liminf_{t\to 0} \frac{\log \operatorname{vol}(D+tA)}{\log m} = 1,$$
$$\limsup_{t\to 0} \frac{\log \operatorname{vol}(D+tA)}{\log m} = \frac{3}{2}.$$

If we are lucky, maybe we can get to h^0 from this. (The possible issue here is that the round-down seems like it it could move you by quite a bit wrt the hyperbolic metric, but I haven't thought it through.)

Theorem 4. *There exists a smooth threefold X with an $\mathbb{R}$-divisor D so that*

$$\limsup_{m\to\infty} \frac{\log h^0(\lfloor mD \rfloor + A)}{\log m} = 2,$$
$$\liminf_{m\to\infty} \frac{\log h^0(\lfloor mD \rfloor + A)}{\log m} = \frac{3}{2}.$$

Roughly speaking, the volume oscillates between m^2- and $m^{3/2}$-type behavior. This shows, in particular, that the invariants κ_σ^+ and κ_σ^- introduced by Nakayama really do not coincide [**?**].

2. Main example

We now consider these quantities in a specific case. Let X be a "Wehler threefold", a hypersurface in $(\mathbb{P}^1)^4$ of type $(2,2,2,2)$. According to Bertini's theorem and the adjunction formula, a general such X is a smooth Calabi–Yau threefold. The Picard rank of this variety is 4, with the real vector space $N^1(X)$ spanned by the pullback to X of $\mathcal{O}_{\mathbb{P}^1}(1)$ from each of the four factors.

The most salient feature of X is that it has a large group of pseudoautomorphisms. For each of the four projections $\pi_i : X \to (\mathbb{P}^1)^3$ (where $1 \le i \le 4$), there is an associated covering involution $\tau_i : X \dashrightarrow X$. The relevant geometry of Wehler threefolds has been investigated in detail by Cantat and Oguiso, who provide a precise picture of the group $\operatorname{Bir}(X)$ and its action on $N^1(X)$. The action of the pullback $\tau_i^* : N^1(X) \to N^1(X)$ is given with respect to our basis by matrices of the form

Theorem 5 (Oguiso–Cantat [**?**]). *The pullbacks τ_i^* are given by*

$$\tau_1^* = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \ldots, \tau_4^* = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Algorithm 1. *Suppose that D is a divisor on X. To compute the*

3. Geodesics

The full birational automorphism group $\operatorname{Bir}(X)$ preserves a quadratic form on $N^1(X)$, which is given with respect the this basis by

$$q = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}.$$

As the signature of this form is $(1, 3)$, the subset $\Delta \subset N^1(X)$ given by

$$\Delta = \left\{ v \in N^1(X) : Q(v, v) = 1 \right\}$$

can be identified with a hyperbolic 3-space $\mathbb{H}^3$, and the action of $\operatorname{Bir}(X)$ on $N^1(X)$ restricts to an action on Δ. Let $\Sigma = \Delta / \operatorname{Bir}(X)$ be a hyperbolic 3-manifold obtained as the quotient; the ample classes $A_\Delta \subset \Delta$ give a fundamental domain. Since $\operatorname{Bir}(X)$ also preserves the volume function, the volume also induces a function $\nu : \Sigma \to \mathbb{R}$.

Now, suppose that D is a pseudoeffective class in $N^1(X)$. Then D lies on the boundary of Δ, so that $Q(D, D) = 0$. If $A = (1, 1, 1, 1)$ is an ample class, then one readily checks that $\alpha = 2\,Q(D, A) \neq 0$ and $\beta = Q(A, A) \neq 0$.

Fix a divisor D on the pseudoeffective boundary and fix an ample divisor A. We are interested in the volumes of the divisors $D(t) = D + tA$. To understand the behavior of the volume as t approaches 0, we normalize $D(t)$ to lie on the hyperbolic space Δ, by setting:

$$\gamma(t) = \frac{D(t)}{\sqrt{Q(D(t), D(t))}} = \frac{D + tA}{\sqrt{t(\alpha + \beta t)}}.$$

(I didn't normalize to constant speed)

Let $\delta(t)$ be the image of $\gamma(t)$ in $\Sigma = \Delta / \operatorname{Bir}(X)$.

We then have

$$\begin{aligned} \operatorname{vol}(D + tA) &= \operatorname{vol}\left(\sqrt{t(\alpha + \beta t)}\gamma(t)\right) = (t(\alpha + \beta t))^{3/2} \operatorname{vol}(\gamma(t)) \\ &= (t(\alpha + \beta t))^{3/2}\, \nu(\delta(t)). \end{aligned}$$

Here $\delta(t)$ is a geodesic which wanders around the hyperbolic three-manifold Σ, while ν is a positive, continuous function on Σ which tends to 0 near the cursps. As a result, when δ is a geodesic which remains away from the cusps (i.e. for which $d(D + tA, A_0)$ is bounded above, $\nu(\delta(t))$ is bounded below and above, so that

$$C_1 t^{3/2} \leq \operatorname{vol}(D + tA) \leq C_2 t^{3/2}$$

for some positive constants C_1 and C_2.

The interesting case is that in which the geodesic $\delta(t)$ approaches the cusps. Since the volume function (of normed classes) approaches ∞ near the cusps, a geodesic which approaches a cusp will have "unexpectedly large" volume for those values of t at which it is on the cusp. We next obtain some precise estimates of the behavior of $\delta(t)$ near a cusp.

The cusps correspond to divisor classes of the form $N_{ij} = H_i + H_j$, which satisfy $q(N_{ij}) = 0$. These give rise to six cusps on Σ.

4. References